# Predicate Logic

## *The Semantic Foundations of Logic*

Richard L. Epstein

**Advanced Reasoning Forum**

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# Contents


## I Propositions and Propositional Logic

A. Logic . . . 2
B. Propositions
1. Propositions and agreements . . . 2
2. Other views of propositions . . . 4
C. Words and Propositions as Types . . . 5
D. Propositions in English . . . 7
• Exercises for Sections A–D . . . 8
E. The Basic Connectives of Propositional Logic . . . 9
F. A Formal Language for Propositional Logic
1. Defining the formal language . . . 11
2. Realizations: semi-formal English . . . 12
• Exercises for Sections E and F . . . 14
G. Classical Propositional Logic
1. *The Classical Abstraction* and truth-functions . . . 15
2. Models . . . 20
3. Validity and semantic consequence . . . 20
4. Determining whether a wff is a tautology . . . 22
H. Examples of Formalization . . . 23
• Exercises for Sections G and H . . . 27
J. Relatedness Logic . . . 28
1. The subject matter of a proposition
a. Relatedness relations . . . 29
b. Subject matter as the content of a proposition . . . 31
2. Models . . . 32
K. An Overview of Semantics for Propositional Logics . . . 35
• Exercises for Sections J and K . . . 38

## II The Internal Structure of Propositions

A. Things, the World, and Propositions . . . 39
B. Names and Predicates . . . 42
C. Propositional Connectives . . . 44
D. Variables and Quantifiers . . . 45
E. Compound Predicates and Quantifiers . . . 47
F. The Grammar of Predicate Logic . . . 48
• Exercises . . . 49

# III A Formal Language for Predicate Logic


A. A Formal Language . . . . . 50
B. The Unique Readability of Wffs . . . . . 53
C. The Complexity of Wffs . . . . . 55
D. Free and Bound Variables . . . . . 56
E. The Formal Language and Propositions . . . . . 58
• Exercises . . . . . 59


# IV Semantics


A. Syntax vs. Semantics as a Basis for Logic . . . . . 63
B. Atomic Propositions . . . . . 64
C. Names
    1. A name picks out at most one thing . . . . . 65
    2. A name picks out at least one thing . . . . . 65
D. Predicates
    1. A predicate applies to an object . . . . . 68
    2. Predications involving relations . . . . . 71
    3. Other conceptions of predicates and predications . . . . . 74
    4. How many predicates are there? . . . . . 76
E. Naming, Pointing, and What There Is
    1. Agreements . . . . . 77
    2. Naming, pointing, and descriptions . . . . . 79
    3. Avoiding names completely? . . . . . 81
    4. Forms of pointing: what there is . . . . . 81
• Exercises for Sections A–E . . . . . 84
F. The Universe of a Realization . . . . . 87
G. *The Self-Reference Exclusion Principle* . . . . . 90
H. Models
    1. The assumptions of the realization: *Form and Meaningfulness* . . 92
    2. Interpretations: assignments of references and valuations . . . . 94
    3. *The Fregean Assumption* and *The Division of Form and Content* . 98
    4. The truth-value of a complex proposition . . . . . 99
    5. Truth in a model . . . . . 104
J. Logics, Validity, Semantic Consequence . . . . . 108
• Exercises for Sections F–J . . . . . 114
*Summary* Chapters II–IV.J . . . . . 116
K. Tarski's Definition of Truth . . . . . 117
    1. Eliminating semantic terms: Convention T . . . . . 118
    2. Other logics, other views of truth . . . . . 122
L. Extensionality
    1. Intensional predicates . . . . . 123
    2. *The Extensionality Restriction* . . . . . 125

3. Quantification and intensional predicates
a. Languages without names . . . 127
b. Models in which every object is named . . . 128
c. Inconsistent predications and quantification . . . 128
M. Other Interpretations of the Quantifiers and the Use of Variables
1. A current variation on Tarski's definition . . . 129
2. The substitutional interpretation . . . 129
3. Naming all elements of the universe at once . . . 131
4. Surveying all interpretations of the name symbols . . . 132
• Exercises for Sections K–M . . . 133

V The Logical Form of a Proposition

A. Rewriting English Sentences . . . 137
B. Common Nouns as Subject and Object
1. Relative quantification: ∀ . . . 139
2. Relative quantification: ∃ . . . 142
3. *Nouns into Predicates* . . . 144
C. Adjectives . . . 145
D. Indexicals . . . 148
E. Adverbs . . . 149
F. Tenses . . . 151
G. Collections and Qualities . . . 154
H. Mass Terms . . . 157
J. Aristotelian Logic . . . 159
K. Formalizations Relative to Formal Assumptions
1. Analysis vs. formalization . . . 162
2. Extending the scope of predicate logic . . . 163
3. Formalizing a notion . . . 164
L. *Criteria of Formalization* . . . 165
M. Examples of Formalization . . . 172
• Exercises . . . 201

VI Identity

A. Identity . . . 208
B. The Equality Predicate . . . 210
C. The Interpretation of '=' in a Model . . . 211
D. The Identity of Indiscernibles
1. *The Predicate Logic Criterion of Identity* (*p.l.c.i.*) . . . 213
2. The *p.l.c.i.* vs. the implicit identity of the universe . . . 215
3. The *p.l.c.i.* and names . . . 216
4. Validity . . . 217
E. Is the Equality Predicate Syncategorematic? . . . 221
• Exercises . . . 223

# VII Quantifiers


A. The Order of Quantifiers
   1. $\forall x \exists y$ and $\exists y \forall x$ . . . 225
   2. $\forall x \exists y$ and $\exists x \exists y$ . . . 226
   3. Superfluous quantifiers . . . 226
B. The Scope of Quantifiers: Substituting One Variable for Another . . . 227
C. Names, Quantifiers, and Existence . . . 230
D. Is '— exists' a Predicate? . . . 232
E. Quantifying Over a Finite Universe: $\forall$ as Conjunction, $\exists$ as Disjunction 233
F. Modeling Other Quantifiers
   1. Positive quantifiers: 'there are at least *n*' . . . 234
   2. Negative quantifiers: 'there are at most *n*', 'no', 'nothing' . . . 236
   3. Exact quantifiers: 'there are exactly *n*' . . . 237
   4. Quantifications we can't model . . . 238
G. Relative Quantification
   1. *Nouns into Predicates* revisited . . . 239
   2. Formalizations involving the same quantifier . . . 240
   3. Formalizations involving mixtures of quantifiers . . . 242
H. Examples of Formalization . . . 243
• Exercises . . . 257


# VIII Descriptive Names


A. Descriptive Names: A Problem in Formalization . . . 263
B. Descriptive Names Relative to Formal Assumptions . . . 265
C. Russell's Method of Eliminating Descriptive Names from Atomic Propositions . . . 266
D. Eliminating All Names? . . . 269
E. Examples of Formalization . . . 272
• Exercises . . . 279


# IX Functions


A. Name-Makers . . . 282
B. Functions
   1. A definition . . . 285
   2. Terms . . . 286
   3. The value of a function . . . 286
   4. Functions compared to predicates . . . 288
C. A Formal Language with Function Symbols and Equality . . . 289
D. Realizations and Truth in a Model . . . 291

E. Partial Name-Makers
1. Russell's abstraction operator . . . 293
2. The ε-operator . . . 297
F. Examples of Formalization . . . 298
• Exercises . . . 300

## X Quantifying Over Predicates: Second-Order Logic

A. Quantifying over Predicates? . . . 304
B. Predicates and Things . . . 305
C. Predicate Variables and their Interpretation: Avoiding Self-Reference
1. Predicate variables . . . 306
2. The interpretation of predicate variables . . . 308
3. Note: Higher-order logics . . . 311
D. A Formal Language for Second-Order Logic: $L_2$ . . . 312
E. Realizations . . . 313
F. Identifying Predicates with Collections of *n*-tuples of the Universe . . . 317
• Exercises for Sections A–F . . . 318
G. Models . . . 319
H. Examples of Formalization . . . 322
• Exercises for Sections G and H . . . 334
J. Predicates as Things: Reducing General Second-Order Logic to First-Order Logic
1. One universe for predicates and individuals . . . 335
2. The translation . . . 338
3. Proof that the mapping preserves consequences . . . 338
4. Does the reduction preserve meaning? . . . 344
K. Quantifying over Functions
1. Why quantify over functions? . . . 345
2. A formal language: $L_{2F}$ . . . 346
3. Realizations and models . . . 347
4. The difficulty of reducing quantification over functions to first-order logic . . . 349
L. Many-Sorted Languages . . . 350
• Exercises for Sections J–L . . . 352

## XI Language, the World, and Predicate Logic

A. The World . . . 353
B. The Template Analogy . . . 353
C. Eliminating Natural Languages? . . . 354
D. Predicate Logic as a Model of or Guide to Reasoning . . . 355

*Appendices*

*A* **The Notion of *Thing* in Predicate Logic** . . . . . 357

*B* **What There Is: Restrictions on the Universe of a Realization** . . . . . 362

*C* **Primitives and Assumptions of Predicate Logic** . . . . 363

*D* **Formalization: Criteria and Agreements** . . . . . 369

*Bibliography* . . . . . 376

*Index of Examples* . . . . . 381

*Index of Notation* . . . . . 388

*Index* . . . . . 390

## *The Semantic Foundations of Logic*

Contents of other Volumes

**Propositional Logics** *Third Edition* published by ARF


I The Basic Assumptions of Propositional Logic
II Classical Propositional Logic
III Relatedness Logic: The Subject Matter of a Proposition
IV Dependence Logics
V Modal Logics
VI Intuitionism
VII Many-Valued Logics
VIII Some Paraconsistent Logics
IX A Temporal Propositional Logic
X A General Framework for Semantics for Propositional Logics
XI Translations Between Logics
XII The Semantic Foundations of Logic


**Classical Mathematical Logic** published by Princeton University Press


I Propositions and Propositional Logic
II Abstracting and Axiomatizing: Classical Propositional Logic
III The Language of Predicate Logic
IV The Semantics of Classical Predicate Logic
V Substitutions and Equivalences
VI Equality
VII Examples of Formalization
VIII Functions
IX The Abstraction of Models
X Axiomatizing Classical Predicate Logic
XI The Number of Objects in the Universe of a Model
XII Formalizing Group Theory
XIII Linear Orderings
XIV Second-Order Classical Predicate Logic
XV The Natural Numbers
XVI The Integers and Rationals
XVII The Real Numbers
XVIII One-Dimensional Geometry
XIX Two-Dimensional Euclidean Geometry
XX Translations within Classical Logic
XXI Classical Logic with Nonreferring Names
XXII The Liar Paradox
XXIII On Mathematical Logic and Mathematics


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To Juney, in loving memory

'Come, let me sniff your soul'

Arf

# Preface

If logic is the right way to reason, why are there so many logics?

In *Propositional Logics* I showed that the forms of reasoning we use depend on what we pay attention to in our reasoning. There is a fundamental unity to the structure of our reasonings. Various logics arise within a general framework of semantics that depend on which, if any aspect of propositions we pay attention to in addition to truth-value, for example, ways in which the proposition could be true, or information content, or subject matter, or ways in which we could come to know the truth-value of the proposition. Propositional logic—reasoning with propositions as wholes, ignoring their internal structure—is summarized in Chapter I.

But what if we consider the internal structure of propositions? Is there still a unity of logics?

In this volume I look at the standard way that logicians parse the internal structure of propositions, what is called *predicate logic*. Most propositional logics have or can be given extensions in the richer language of predicate logic. My goal here is to find or suggest agreements about language, the world, and reasoning that can account in a uniform way for those extensions.

My focus in describing those agreements will be in viewing logic as a form of reasoning in language. Logic, I believe, can be understood best as something we do, whether it be prescriptive or descriptive. The agreements we make about our forms of reasoning seem based on ideas about the relation of language and the world which, if ignored, lead only to empty symbolism. Those agreements, I stress, need not be conventions. They may arise from the nature of the world, or the way our bodies are built, or the reality of abstract forms, or . . . . I am not able to say. But what I can do here is give a story that seems to me to unify the various stories of many logicians.

I had at one time hoped to give technical developments of many predicate logics in this volume. That, it has turned out, is too large a task, and I have chosen to defer the technical development of particular predicate logics to other volumes, whose contents I describe at the end of the table of contents.

For a philosopher, the project of this volume may seem commonplace: an account of propositions and semantics. For a mathematician or computer scientist it may seem less clear why this concern is important. But understanding the basis of predicate logic is essential for formalizing ordinary language reasoning, which is the heart of artificial intelligence and every use of computers to glean information from what we present to them. Moreover, the uniform treatment of all predicate logics makes it clearer when a logic is or is not suitable for a particular task of formaliza-

tion, and allows for the construction of other logics to codify reasoning in contexts outside the scope of current logics.

Briefly, the contents of this volume are as follows:

Chapter I is devoted to a summary of reasoning with propositions as wholes, propositional logic.

In Chapter II the predicate logic way of parsing propositions into predicates and names is introduced. A formal language in Chapter III gives structure and precision to that way of parsing.

Chapter IV connects reasoning in language to ideas about the world and the nature of truth, giving a semantic analysis of reasoning with predicate logic forms. The notion of a model and realization are developed in such a way as to serve as the basis for a spectrum of logics, without specifying any one particular logic from that spectrum.

In Chapter V I turn to the issue of how to formalize ordinary reasoning in predicate logic. To formalize is to translate from our daily language to the language of predicate logic. Formalizations, then, should satisfy the standards of a good translation: They should preserve inferences, respect form when possible, preserve meaning. I present criteria for judging a formalization in Chapter V that are tested and that shape the work in the succeeding chapters.

Chapter VI deals with how to formalize "is the same as" in predicate logic. Chapter VII adopts standards for formalizing propositions that use ordinary language quantifications. In Chapter VIII the criteria of formalizing force a decision about how we shall formalize propositions containing descriptive names. Chapter IX then deals with how to incorporate functions in the language of predicate logic.

Chapter X considers parsing propositions to account for further structure in our reasoning, quantifying over not only individual things but also collections or predicates of those things. Disputes about the nature of predicates do not affect the basis of predicate logic, since predicates are used only to the extent that they can be identified with pieces of language. However, in an analysis of reasoning that allows for quantifying over predicates, disagreements are sufficient to generate no common logic. Whether the collection of all predicates contains objects that cannot be identified with pieces of language affects the forms of reasoning that are deemed valid.

The presentation of the ideas in each chapter begins with a few examples as motivation to lead to an exposition of general principles. Those principles are then examined in a series of examples that are discussed in detail in an example/analysis format. Some examples are meant only to illustrate the meaning or use of a principle, but many advance the theory.

In the final chapter and first appendix I summarize the views that have motivated this development of logic, trying to explain further why they seem to me to make a good story. I hope to have given a good story, one that not only accounts for why there are so many logics, but also answers: Why predicate logic? Still, I do

not claim that this is the only story, nor that it is especially original: The literature on predicate logic is vast, and I regret that I cannot trace here the history of the subject nor give adequate space to other stories of how we should reason.

I have included many exercises, some of them routine and some requiring considerable thought. There is no instructor's manual, for often there may be more than one good answer, more than one way to argue a point or formalize a proposition, and those differences will, I hope, stimulate you to come to your own views on the nature of logic.

* * * * * * * * *

Since the first publication of this work in 1994 I have extended my research to much of the traditional scope of logic and have shown how to expand the scope of what can be formalized in predicate logic. The publications in which I have done that are listed in the bibliography, and most of that work is now available from the Advanced Reasoning Forum, <www.Advanced ReasoningForum.org>. It would be too large a project for me to try to modify this text to take account of that now. There is one point, though, that should be noted: Where I talk here of arguments or deductions in formal logic I believe now that it is better to speak of inferences. I have corrected a few typographical errors in the previous edition.

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